Proof of the causal model

We need to calculate $p(\mathbf{e}|\mathbf{x}, \mathbf{c})$. Applying Bayes' theorem, we have that

$$p(\mathbf{e}|\mathbf{x}, \mathbf{c}) = \frac{p(\mathbf{x}|\mathbf{e}, \mathbf{c})p(\mathbf{e}|\mathbf{c})}{p(\mathbf{x}|\mathbf{c})}. \tag{1}$$

If the causal model is captured by the Directed Acyclic Graph (DAG) $\mathbf{c} \to \mathbf{e} \to \mathbf{x}$, we also have that

$$p(\mathbf{x}|\mathbf{e}, \mathbf{c}) = p(\mathbf{x}|\mathbf{e}). \tag{2}$$

Applying Bayes' theorem again, we have that

$$p(\mathbf{x}|\mathbf{e}) = \frac{p(\mathbf{e}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{e})}. \tag{3}$$

Thanks to equations 2 and 3, we can substitute $p(\mathbf{x}|\mathbf{e}, \mathbf{c})$ in equation 1 with $\frac{p(\mathbf{e}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{e})}$ and write

$$p(\mathbf{e}|\mathbf{x}, \mathbf{c}) = \frac{p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c})p(\mathbf{x})}{p(\mathbf{x}|\mathbf{c})p(\mathbf{e})}. \tag{4}$$

Now, because $p(\mathbf{e}|\mathbf{x}, \mathbf{c})$ is a probability, we know that

$$\int_{\mathbf{e}} p(\mathbf{e}|\mathbf{x}, \mathbf{c})d\mathbf{e} = 1. \tag{5}$$

But, because of equation 4, this also implies that

$$\int_{\mathbf{e}} \frac{p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c})p(\mathbf{x})}{p(\mathbf{x}|\mathbf{c})p(\mathbf{e})}d\mathbf{e} = 1. \tag{6}$$

We can observe that $p(\mathbf{x})$ and $p(\mathbf{x}|\mathbf{c})$ are constant with respect to $\mathbf{e}$, therefore they can be taken outside the integral:

$$\frac{p(\mathbf{x})}{p(\mathbf{x}|\mathbf{c})}\int_{\mathbf{e}}\frac{p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c})}{p(\mathbf{e})}d\mathbf{e} = 1. \tag{7}$$

In the next step, we use the assumption that the prior $p(\mathbf{e})$ is uniform. If the $p(\mathbf{e})$ is uniform, it does not depend on $\mathbf{e}$, therefore, we can take it outside the integral too:

$$\frac{p(\mathbf{x})}{p(\mathbf{x}|\mathbf{c})p(\mathbf{e})}\int_{\mathbf{e}}p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c})d\mathbf{e} = 1. \tag{8}$$

Multiplying on both sides by $\frac{p(\mathbf{x}|\mathbf{c})p(\mathbf{e})}{p(\mathbf{x})}$, we obtain that

$$\int_{\mathbf{e}}p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c})d\mathbf{e} = \frac{p(\mathbf{x}|\mathbf{c})p(\mathbf{e})}{p(\mathbf{x})}. \tag{9}$$

At this point, we can substitute $\frac{p(\mathbf{x}|\mathbf{c})p(\mathbf{e})}{p(\mathbf{x})}$ in equation 4 with the integral on the left hand side of equation 9, obtaining

$$p(\mathbf{e}|\mathbf{x},\mathbf{c}) = \frac{p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c})}{\int_{\mathbf{e}}p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c})d\mathbf{e}}. \tag{10}$$

Note that $\int_{\mathbf{e}}p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c})d\mathbf{e}$ is integated over all values of $\mathbf{e}$, and therefore it is constant with respect to changes in $\mathbf{e}$. Thus, in conclusion:

$$p(\mathbf{e}|\mathbf{x},\mathbf{c}) \propto p(\mathbf{e}|\mathbf{x})p(\mathbf{e}|\mathbf{c}). \tag{11}$$